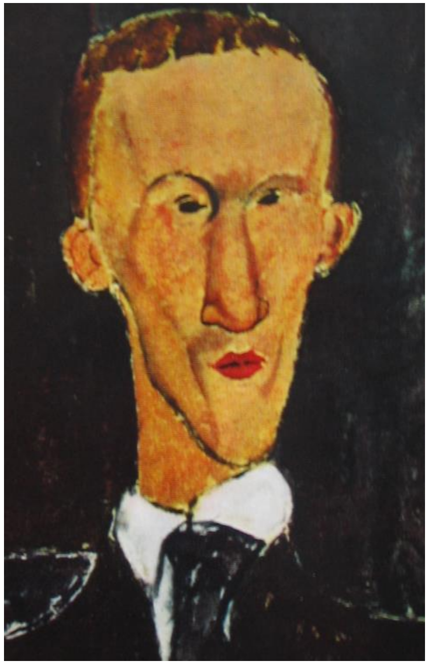

modigliani

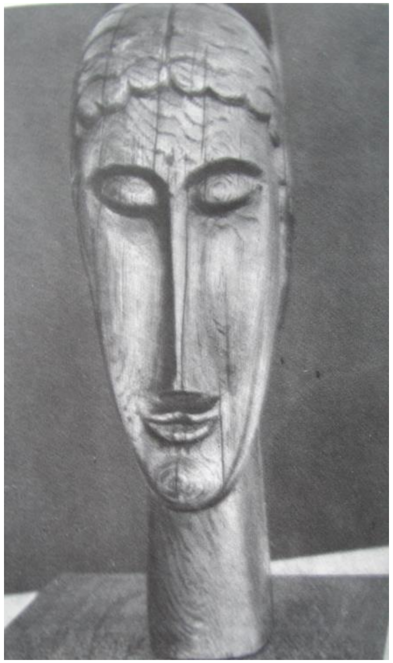
EU-Italy-Russia@Dubna Round Table

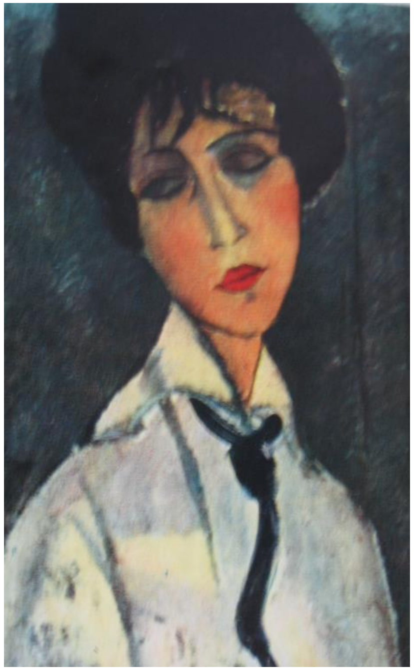
Dubna, 3-5 March 2014

A.T. FILIPPOV (JINR, Dubna)

MANY FACES OF SCALARON







Early short history of scalarons (scalar *fields* related to gravity)

Kaluza 1919; Mandel, Klein, Fock 1926; Einstein, Bergmann 1938;

Jordan 1959; Brans, Dicke 1961; G.S.W.- NNB 1961; HBE-GHK 1964;

Reductions of HD Yang-Mills-Gravity theories (in eighties…)

Superstrings , Supergravity : reductions from High Dimensions
(using generalizations of Kaluza – Einstein Bergmann ideas)

Systematic studies of various Dilaton Gravities (nineties…)
(Integrable models, solitons, dynamical chaos…)

Nonlinear dynamicas of static, cosmological and wave solutions in DG

(Sclarons in cosmology is a separate long and complicated story)

Content of the talk

Brief discussion of models based on Superstrings, SUGRA, and scalar particles coupled to gravity.

Very brief summary of **affine models**
Vecton – Scalaron equivalence in Dilaton Gravity for $D = 2,1$

Dilaton Gravity coupled to **scalars** (**DGS**).
Unified treatment of **static** (BH) and **cosmological** solutions
Dynamical systems of static & cosmological states in DGS

Integrability *vs*. Nonintegrability: new integrable models etc.

Partially integrable DSG (insufficient # of integrals) and approximate methods for BHs and cosmologies (cosmostat)

The Aim: Topological Portraits (here only idea and simplest examples)

see: Modigliani, Sutin e.a.

See also: arXiv:1302.6372, 1112.3023 (ATF) ; 1302.6969 (EAD+ATF); + references therein

$$\sqrt{-g^{(10)}}\left(e^{-2\phi^{(10)}}R^{(10)} + 4e^{-2\phi^{(10)}}(D\phi^{(10)})^2 - \frac{1}{12}H^{(10)2}\right)$$

10-D SUGRA
is dim. reduced
to dimension 6

$$ds^2 = g^{(6)}_{\mu\nu}dx^\mu dx^\nu + e^\psi dx^m dx^m$$

$$H^{(10)} = dB$$

NS - 3-form

$$g^{(6)}_{\mu\nu} = \begin{pmatrix} g^{(5)}_{\alpha\beta} + e^{\psi_1}A_\alpha A_\beta & e^{\psi_1}A_\alpha \\ e^{\psi_1}A_\beta & e^{\psi_1} \end{pmatrix}$$

Kaluza reduction
to 5-brane

$S^1 \times T^4$

and then spherical
reduction to DGS

$$H' = H - A\wedge F_2 \qquad F_{2\alpha\beta} = H_{\alpha\beta x^5}$$

In D=2: $H'_{ijk} = H_0\epsilon_{ijk}$

$$\sqrt{-g}e^{-2\phi}\Big(R + 6e^{2\psi_2} + 4(D\phi)^2 - (D\psi)^2 - \frac{1}{4}(D\psi_1)^2 - 3(D\psi_2)^2$$

$$-\frac{1}{2}e^{6\psi_2}H_0^2 - \frac{1}{4}e^{-\psi_1}F_2^2 - \frac{1}{4}e^{\psi_1}F^2\Big)$$

e.g.,Y.Kiem e.a.
hep-th/9806182

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + e^{-2\psi_2}d\Omega_{(3)}$$

Spherical reduction on 5-brane

Generalized cylindrical reduction of 4-dimensional Einstein gravity
(Kaluza - type reduction)

$$ds_4^2 = (g_{ij} + \varphi\,\sigma_{mn}\,\varphi_i^m \varphi_j^n)\,dx^i dx^j + 2\varphi_{im}\,dx^i dy^m + \varphi\,\sigma_{mn}\,dy^m dy^n$$

x-coordinates (t, r) $\quad\quad$ $y^m = (\varphi, z)$ $\quad\quad$ (coordinates on torus)

$$\sigma_{22} = e^{\eta}\cosh\xi, \quad \sigma_{33} = e^{-\eta}\cosh\xi, \quad \sigma_{23} = \sigma_{32} = \sinh\xi$$

$$\sqrt{-g}\left[\varphi R(g) + \frac{1}{2\varphi}(\nabla\varphi)^2 - \frac{\varphi}{2}[(\nabla\xi)^2 + (\cosh\xi)^2(\nabla\eta)^2] + V_{\text{eff}}(\varphi,\xi,\eta)\right]$$

$$V_{\text{eff}}(\varphi,\xi,\eta) = -\frac{\cosh\xi}{2\varphi^2}\left[Q_1^2 e^{-\eta} - 2Q_1Q_2\tanh\xi + Q_2^2 e^{\eta}\right]$$

Interesting **consistent truncation**: $\quad\quad$ $Q_1 \neq 0,\ Q_2 = 0,\ \xi \equiv 0$:

$$\mathcal{L}_{\text{eff}}^{(2)} = \sqrt{-g}\left[\varphi R(g) + \frac{1}{2\varphi}(\nabla\varphi)^2 - \frac{Q_1^2}{2\varphi^2}\,e^{-\eta} - \frac{\varphi}{2}\,(\nabla\eta)^2\right]$$

Main principles (suggested by Einstein's approach)

1. **Geometry:** **dimensionless** *`action'* constructed of a ***scalar*** *density*; its variations give the geometry and main equations ***without complete specification of the analytic form of the Lagrangian .***

2. **Dynamics:** a concrete Lagrangian constructed of the ***geometric variables*** - homogeneous function of order D (e.g. , the square root of the determinant of the curvature) produces a physical **effective Lagrangian**.

3. **Duality** between the geometrical and physical variables and Lagrangians.

NB: This looks more artificial than the first two principles and, possibly, works for rather special models (actually giving *exotic fields*, *tachyons* etc.)

A simple nontrivial choice of a **geometric Lagrangian** density, which generalizes the Eddington – Einstein Lagrangian ,

$$\mathcal{L} \equiv \sqrt{-\det(r_{ij})} \equiv \sqrt{-r},$$

is the following (depending on one dimensionless parameter):

$$\mathcal{L}_{geom} = \sqrt{-\det(s_{ij} + \bar{l} a_{ij})} \equiv \sqrt{-\Delta_s}$$

It is more convenient to take:

$$\mathcal{L} \equiv 2\tilde{\gamma}\sqrt{-\Delta_s}$$

It is easy to derive the important relation

$$|\Delta_g| \equiv |\det(\mathbf{g}^{ij} + l\,\mathbf{f}^{ij})| = \tilde{\gamma}^D\,|\det(s_{ij} + l^{-1}\,a_{ij})|^{(D-2)/2}\,.$$

and using the relations between geometrical and `physical' variables
to find the complete effective Lagrangian replacing the EH one

$$\mathcal{L}_{phys} = \sqrt{-g}\left[-2\Lambda\,[\det(\delta_i^j + l\,f_i^j)]^\nu + R(g) - m^2\,g^{ij}a_i a_j\right]$$

Λ having the dimension L^{-2}

l is a dimensionless

$$[s_{ij}] = [a_{ij}] = \mathrm{L}^{-2}$$

$$[\tilde{\gamma}] = \mathrm{L}^{D-2}$$

Dimensional reductions of

$$\mathcal{L}_{\rm ph} = \sqrt{-g}\left[-2\Lambda\left[\det(\delta_i^j + \lambda f_i^j)\right]^\nu + R(g) + c_a\, g^{ij} a_i a_j\right]$$

Spherical reduction of the theory

$$ds_D^2 = ds_2^2 + ds_{D-2}^2 = g_{ij}\, dx^i\, dx^j \; + \; \varphi^{2\nu}\, d\Omega_{D-2}^2(k)$$

$$\mathcal{L}_D^{(2)} = \sqrt{-g}\left[\varphi R(g) + k_\nu\, \varphi^{1-2\nu} + \frac{1-\nu}{\varphi}(\nabla\varphi)^2 + X(\varphi, \mathbf{f}^2) - m^2 \varphi\, \mathbf{a}^2\right]$$

$$X(\varphi, \mathbf{f}^2) \equiv -2\Lambda\varphi\left[1 + \frac{1}{2}\lambda^2\mathbf{f}^2\right]^\nu \qquad \mathbf{f}^2 \equiv f_{ij} f^{ij} \qquad \nu \equiv (D-2)^{-1}$$

D=3: Maxwell+Einstein+cosm. constant

Weyl rescaling

$$g_{ij} = \hat{g}_{ij}\, w^{-1}(\varphi), \quad w(\varphi) = \varphi^{1-\nu} \qquad \mathbf{f}^2 = w^2\, \hat{\mathbf{f}}^2, \quad \mathbf{a}^2 = w\, \hat{\mathbf{a}}^2$$

$$\mathcal{L}^{(2)}_{DW} = \sqrt{-g}\left[\varphi R(g) + k_\nu\,\varphi^{-\nu} - 2\Lambda\varphi^\nu\left[1+\frac{1}{2}\lambda^2\varphi^{2(1-\nu)}\mathbf{f}^2\right]^\nu - m^2\varphi\,\mathbf{a}^2\right]$$

3-dimensional theory

$$\mathcal{L}^{(2)}_{3} = \sqrt{-g}\left[\varphi R(g) - 2\Lambda\varphi - \lambda^2\Lambda\varphi\,\mathbf{f}^2 - m^2\varphi\,\mathbf{a}^2\right]$$

Vecton – Scalaron DUALITY

in LC coordinates

$$ds^2 = -4\,h(u,v)\,du\,dv\,, \qquad \sqrt{-g} = 2h \qquad\qquad f^n_{uv} \equiv a^n_{u,v} - a^n_{v,u}$$

L /2*h* = $\varphi R + V(\varphi,\psi) + X(\varphi,\psi;\mathbf{f}^2_n)$ $\qquad -2\mathbf{f}^2_n = (f^n_{uv}/h)^2$

L' /2*h* = $\varphi R + V(\varphi,\psi) + X_{\rm eff}(\varphi,\psi;q_n)$ $\qquad q_n(u,v) \equiv h^{-1}X_n\,f^n_{uv}$

$$X_n \equiv \frac{\partial X}{\partial\,\mathbf{f}^2_n}$$

The result: we can study **DSG** instead of **DVG**

$$X_{\mathrm{eff}} \quad = -2\Lambda\sqrt{\varphi}\Big[1 + q^2/\lambda^2\Lambda^2\varphi^2\Big]^{\frac{1}{2}}$$ for $D = 4$

$$V = 2k\varphi^{-\frac{1}{2}}\,, \quad \bar{Z} = -1/m^2\varphi$$

N.B: normally, ***Z*** ~ to dilaton φ

$$X_{\mathrm{eff}}(\varphi;\; q(u,v)) = -q^2/\lambda^2\Lambda\varphi - 2\Lambda\varphi$$ for $D = 3$

General **dilaton gravity** coupled to massless vectors and eff. massive scalars

$$\mathcal{L}^{(2)} = \sqrt{-g}\,[\,U(\varphi)R(g) + V(\varphi) + W(\varphi)(\nabla\varphi)^2 +$$
$$+X(\varphi, \psi, F^2_{(1)}, ..., F^2_{(A)}) + Y(\varphi, \psi) + \sum_n Z_n(\varphi, \psi)(\nabla\psi_n)^2\,]\,.$$

Dilaton gravity **dual** to vecton gravity with **massless Abelian vector fields,** Weyl frame

$$\mathcal{L}^{(2)}_{\mathrm{eff}} = \sqrt{-g}\left[\varphi R + V(\varphi, \psi) + X_{\mathrm{eff}}(\varphi, \psi; q) + \sum Z(\varphi, \psi)(\nabla\psi)^2\right].$$

DSG dual to **massive** vecton gravity (Weyl frame) has kinetic q-term

$$\mathcal{L}_{\mathrm{dsg}} = \sqrt{-g}\left[\varphi R + U(\varphi, \psi, q) + \bar{Z}(\varphi)(\nabla q)^2 + \sum Z(\varphi, \psi)(\nabla\psi)^2\right]$$

A general theory of **HORIZONS** in DSG

L' /2h = $\varphi R + U(\varphi, \psi, q) + \bar{Z}(\varphi)(\nabla q)^2$ (omitting normal scalars)

Consider **STATIC** solutions that normally **have horizons** when there are **no scalars**

All the equations can be derived from the Hamiltonian (constraint)

H = $\dot{\varphi}\,\dot{h}/h + hU + \bar{Z}\,\dot{q}^2 + Z\,\dot{\psi}^2$ (**= 0** in the end)

Without the scalars the EXACT solutions is: $h = C_0^2\,[N_0 - N(\varphi)]$

where $N(\varphi) \equiv \int U(\varphi) d\varphi$ $C_0 \tau = \int d\varphi\,[N_0 - N(\varphi)]^{-1}$

There is always a horizon, i.e. $h \to 0 \text{ for } \varphi \to \varphi_0$

When the potential depends on scalaron ***q*** or on other scalars **a HORIZON exists**

Horizons are classified into:

regular **simple**, regular **degenerate**, **singular**

$$\epsilon\mathcal{L}^{(1)} = -\bar{l}_\epsilon^{-1}\left[\dot{\psi}^2 + 2\dot{\alpha}_\epsilon\,\dot{\xi}\,\right] + \bar{l}_\epsilon\,\epsilon\, e^{2\alpha_\epsilon}\, U(\xi,\psi)$$

Static-Cosm. Lagr.

$$d\,\xi \equiv Z^{-1}(\varphi)\,d\varphi \qquad U(\xi,\psi) \equiv Z(\varphi)V(\varphi,\psi)$$

Equations and Constraint

$$\ddot{\xi} + h\,U = 0\,, \quad 2\,\ddot{\psi} + h\,U_\psi = 0\,, \qquad F \equiv \ln|h| \equiv 2\alpha_\epsilon,$$

$$\ddot{F} + h\,U_\xi = 0\,; \qquad \dot{\psi}^2 + \dot{F}\dot{\xi} + h\,U = 0\,.$$

Integrable DG (ATF 1996)

$$2\lambda_1 U \;=\; g_+\exp(\lambda_1\xi) \;-\; g_-\exp(-\lambda_1\xi) \qquad \psi_1 \equiv (F-\xi)/2,\ \psi_2 \equiv (F+\xi)/2,$$

$$V = g_1\exp(2\lambda_1\psi) + g_2\exp(2\lambda_2\psi),\ Z = -\varphi \equiv -e^{-\xi}$$

Simplest Liouville DG

General *N*-Liouville (next generalization: Toda – Liouville, VdA + ATF)

$$\mathcal{L} = -\bar{l}^{-1}\sum_1^N \varepsilon_n\dot{\psi}_n^2 - \bar{l}\sum_1^N g_n e^{q_n}\,, \qquad q_n \equiv \sum_1^N \psi_m a_{mn}\,,$$

$$\hat{A} \equiv \hat{a}^T\hat{\varepsilon}\,\hat{a},\ \text{where } \hat{\varepsilon}_{mn} \equiv \varepsilon_m\delta_{mn}$$

$U = g\,\xi + v(\psi)$ Interesting integrals for polynomial ***V***

$$\psi''(F) = -\gamma p\, e^F \psi^{p-1}\,, \quad \xi''(F) = -\gamma\, e^F \psi^p\,; \qquad \text{for} \qquad v = \bar{g}\,\psi^p$$

$U = g$ The simplest portrait

$$\delta \equiv C_2/C_1\,, \quad C_3^2 + C_1 C_2 = 0\,,$$

$$\xi - \xi_0 = -\hat{h} + \ln|h|^{\delta}\,; \qquad \hat{h} \equiv h/h_0\,, \quad h_0 \equiv C_1^2/g$$

For p =1: The portrait is rather interesting:

$$\psi = -\gamma\,(e^F + C_1 F + C_2)\,, \quad \xi - \xi_0 = \gamma^2\,[\,e^{2F}/\,4 + (C_2 - 2C_1)\,e^F + C_1 F e^F - C_1^2\,F/\,2\,]\,;$$

If $v = \bar{g}^p$ and $p \neq 0, 1, 2,$ we have **Emden – Fowler** equation

which can be reduced to the **autonomous equation**:

$$z'' + z' + z - z^{p-1} = 0\,, \qquad (z^{p-1} - z - 2\,y)\,dy = y\,dz\,.$$

One can find a symmetry and the corresp. integral

We find a gen. sol. near horizon as locally convergent power series in:

$$h = \sum h_n \tilde{\varphi}^n, \qquad \chi = \sum \chi_n \tilde{\varphi}^n, \qquad q = \sum q_n \tilde{\varphi}^n, \qquad \chi(\varphi) \equiv \dot{\varphi}$$

$$h_0 = \chi_0 = 0 \qquad q_0 \neq 0 \qquad \tilde{\varphi} \equiv \varphi - \varphi_0$$

The equations for these functions are **not integrable** and we do not know exact solutions of the recurrence relations

Practically the **same equations** are applicable to studies of the **cosmological models** with vecton.

However, we can show that the global picture cannot be found without knowledge of horizons connecting static and cosmological solutions.

It is important to use local language. BUT! The physics can not be well understood without global picture – topological «portrait»

Main differential equations

$$\psi' = E(\xi)\,, \quad H' = -E^2 H(\xi)\,;$$

$$\chi' = -Z(\varphi)\,U(\varphi,\psi)\,H\,, \quad \eta' = -\frac{1}{2}Z(\varphi)\,U_\psi(\varphi,\psi)\,H\,.$$

$$\psi' \equiv \frac{d\psi}{\partial\xi}\,, \quad U_\psi \equiv \frac{\partial U}{\partial\psi}\,, \quad \xi \equiv \int d\varphi\, Z^{-1}(\varphi)\,, \quad Z(\varphi) = 1/\xi'(\varphi)\,.$$

$$E(\xi) \equiv \frac{\eta(\xi)}{\chi(\xi)}\,, \quad H(\xi) \equiv \frac{h(\xi)}{\chi(\xi)}\,. \qquad \frac{d\eta}{d\chi} = \frac{U_\psi}{2U}\,, \quad \frac{d\ln H}{d\psi} = -E\,.$$

$$\psi(\xi) = \psi_0 + \int_{\xi_0}^{\xi} E(\bar\xi) \equiv \mathcal{I}\{E;\xi\}\,,$$

Solutions in terms of **one** function ***E***

Basic solutions

$$H(\xi) = H_0 \exp \int_{\xi_0}^{\xi} E^2(\bar\xi) \equiv H_0\, \exp \mathcal{I}\{E^2;\xi\}\,,$$

$$\chi(\xi) = \chi_0 - \mathcal{I}_1\{E;\xi\}\,, \quad \eta = \eta_0 - \mathcal{I}_2\{E;\xi\}\,,$$

$$\mathcal{I}_1\{E;\xi\} = -H_0 \int_{\xi_0}^{\xi} d\bar{\xi}\, Z[\varphi(\bar{\xi})]\; U[\varphi(\bar{\xi}, \mathcal{I}\{E;\bar{\xi}\}]\; e^{\mathcal{I}\{E^2;\bar{\xi}\}}\,.$$

$$\mathcal{I}_2\{E;\xi\} = -\frac{1}{2} H_0 \int_{\xi_0}^{\xi} d\bar{\xi}\, Z[\varphi(\bar{\xi})]\; U_{\psi}[\varphi(\bar{\xi}), \mathcal{I}\{E;\bar{\xi}\}]\; e^{\mathcal{I}\{E^2;\bar{\xi}\}}$$

THE MASTER INTEGRAL EQUATION

$$E(\xi) = \frac{\eta_0 - \mathcal{I}_2\{E;\xi\}}{\chi_0 - \mathcal{I}_1\{E;\xi\}}$$

Extended dynamical system

$$\rho + UH + \eta^2/\chi = \chi' + UH = \eta' + U_\psi H/2 = \rho' + U_\xi H = 0$$

$$\psi\,\eta' + \psi\, U_\psi\, H/2 = \eta\,\psi' - \eta^2/\chi = \xi\,\rho' + \xi\, U_\xi\, H = \xi'\rho - \rho = 0$$

generates integrals

$$F \equiv \ln h$$

$$\chi F' - \rho = 0$$

$$[c_1\chi + c_2\,\eta + c_3\,\rho + c_4(\psi\,\eta + \xi\,\rho)]' =$$

$$-H[(c_1 + c_4)\,U + c_2\,U_\psi/2 + c_3\,U_\xi + c_4(\psi\,U_\psi/2 + \xi\,U_\xi)]$$

$$c_1\chi + c_2\,\eta + c_3\,\rho + c_4(\psi\,\eta + \xi\,\rho) = I_1$$ **= integral**

IF $$(c_1 + c_4)\,U + c_2\,U_\psi/2 + c_3\,U_\xi + c_4(\psi\,U_\psi/2 + \xi\,U_\xi) = 0$$

$$U = U_0 \exp(2g\psi + g_1\xi) \quad \Longrightarrow \quad \chi' = (g^2 + g_1)\,\chi + (\rho_0 + 2\,\eta_0\,g) + \eta_0^2/\chi$$

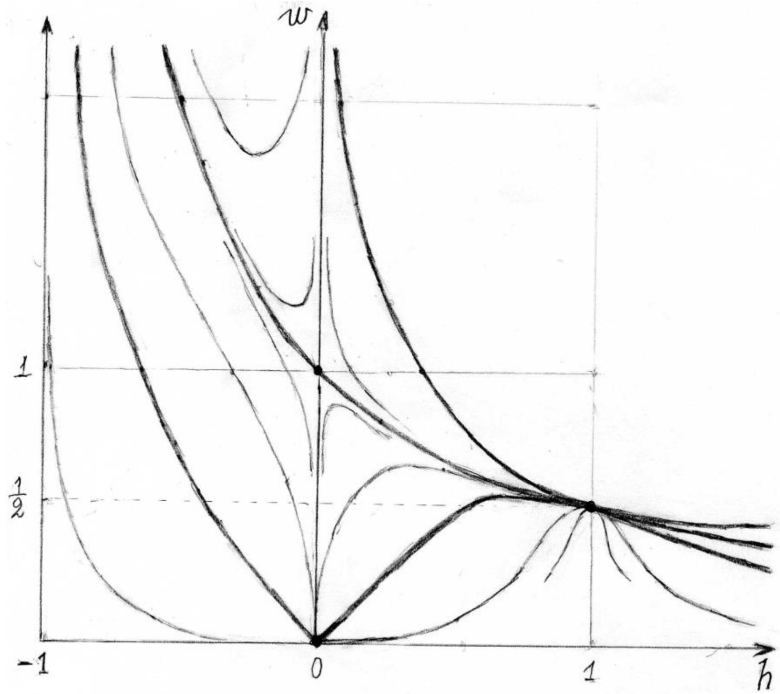
As distinct from the standard Einstein theory, the generalized one is **not integrable** even in dimension one (static states and cosmologies). Therefore, in addition to the above solutions we need a global information on the system, which we may attempt to present as

topological portrait.

We try to demonstrate that the portrait must include **both static and cosmological** solutions, and that the most important info is in the structure of horizons. Actually, it is not less important for cosmologies than for static states. We prefer to use the **local language** and do not use the term Black Hole which should be reserved for real physical objects

For the moment, the idea can be explained only on integrable systems and only on the plane. For nonintegrable systems we need **3D portraites**

Solutions from ATF 1996



$$w = \frac{|h|^{\delta}}{|1 + \varepsilon |h|^{1+2\delta}|}\,.$$

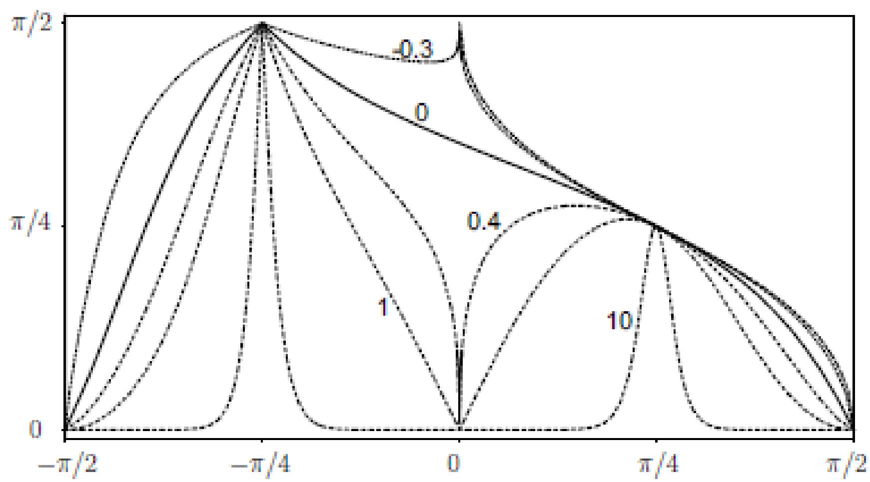
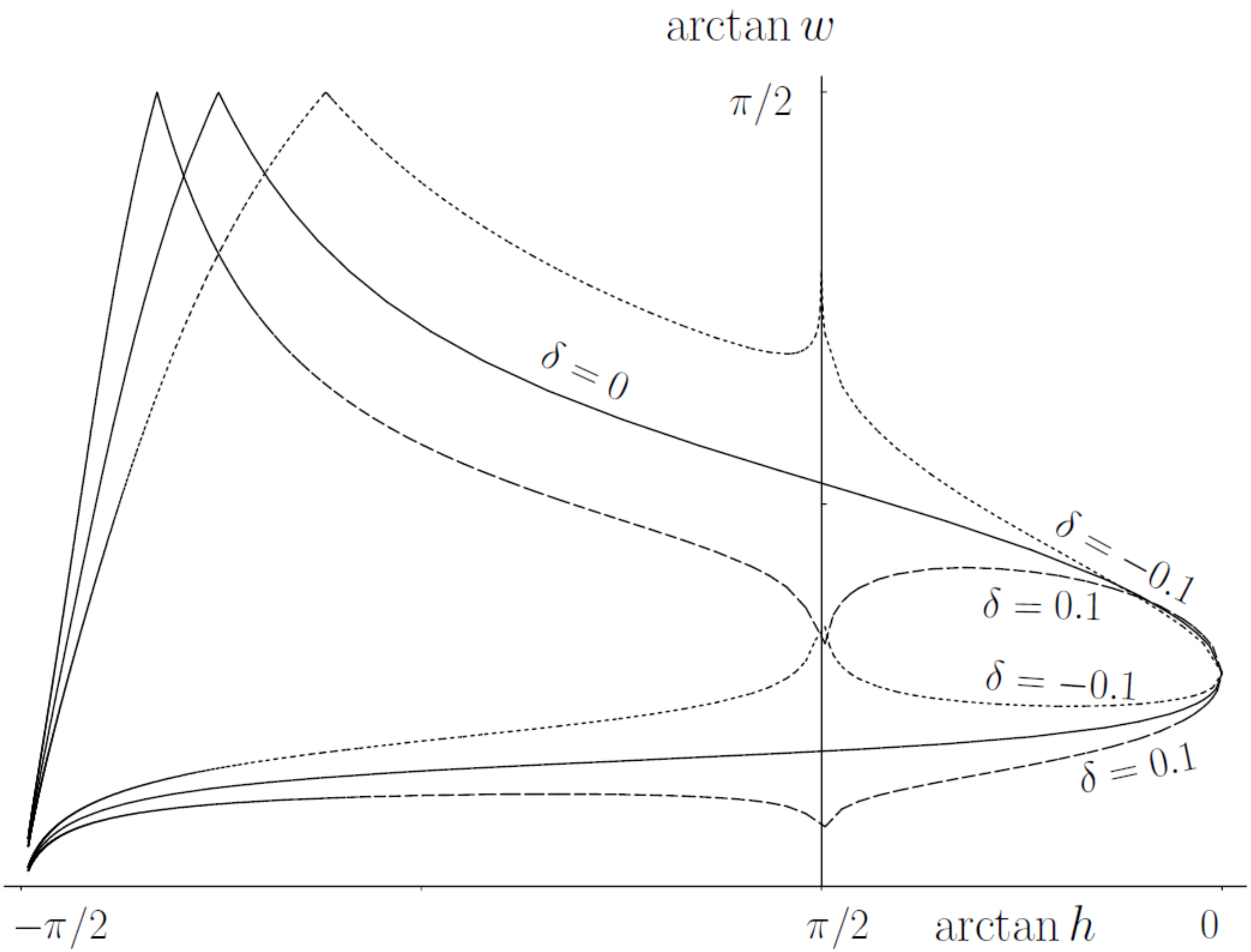


Figure 1: Topological portrait arctan $w(x)$ of the dilaton-scalar configuration. Values of δ are given on plot. Solid line — separatrix with zero δ, dashed line — separatrix with arbitrary δ.

Solutions from ATF 1996



Topological portrait of the dilaton-scalar-vector massless configuration with the cosmological special point $(h = 1, w = 1/2)$

Picture by E.Davydov

In DVG derived from affine theories, we studied *one-dimensional dynamical systems simultaneously describing cosmological and static states.*

Our approach is fully applicable to static, cosmological, and simple wave solutions in multidimensional theories (esp., by using the **scalaron - vecton duality**) and to **general** one-dimensional DGS models.

The global structure of the solutions of integrable DGS-1 models can be usefully visualized by drawing their **`topological portraits'** resembling the phase portraits of dynamical systems and simply visualizing

static – cosmological duality

Бог радости, музыки и танца сапотеков.
Окрашенная глина

A god of joy,
music and dance

THE

END

The Emden-Fowler equation

$$y'' + \frac{a}{x}y' + bx^{m-1}y^n = 0, n \neq 0$$

Transformation to autonomous form

$$\ddot{z} - \frac{(1-a)(n-1)+2(1+m)}{n-1}\dot{z} + \frac{[(1-a)(n-1)+1+m](1+m)}{(n-1)^2}z + bz^n = 0$$

where $$y = x^{(1+m)/(1-n)}z,\ dt = x^{-1}dx$$

$$group\ \ x_1 \ = \ e^{\epsilon}x,\ \ y_1 \ = \ e^{-2\epsilon(1+m)/(n-1)}y,$$

generator $$X = x\frac{\partial}{\partial x} + \frac{1+m}{1-n}y\frac{\partial}{\partial y}.$$

The generalized Emden-Fowler equation

$$y'' + a_1(x)y' + a_0(x)y + f(x)y^n = 0$$

can be reduced to the autonomous form

$$\ddot{z} \pm b_1\dot{z} + b_0 z + c z^n = 0$$

Symmetry in Nonlinear Mathematical Physics 1997, V.1, 155–163.

See idetails in **The Generalized Emden-Fowler Equation**

L.M. BERKOVICH